Lec 2 - Symmetries as Extended Defect Ops

$$S = \int_{M} d^d x \, \mathcal{L}$$

$\mathcal{L}[\phi]$ $\quad \phi \mapsto \phi + \alpha \delta\phi$

$$S \mapsto S + \delta S$$

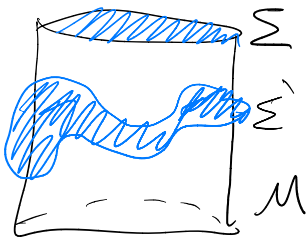
$$\delta S = \int d^d x \, \partial_\mu J^\mu$$

$$\delta S = \int d^d x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \delta x^\mu \right)$$

$$\partial_\mu j^\mu = 0$$

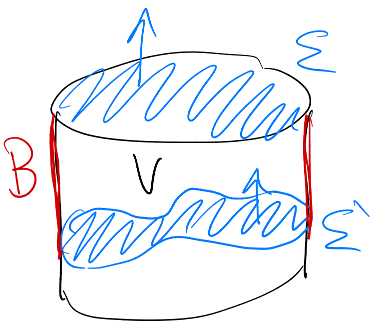
$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \delta x^\mu - J^\mu$$

$$\partial_\mu j^\mu = 0 \qquad d * j = 0$$



$$Q = \int_\Sigma d^{d-1}x \, j^0$$

$$Q(\Sigma) = \int_{\Sigma'} * j$$



$$\partial V = \Sigma - B - \Sigma'$$

$$\int_V d * j = 0$$

$$\parallel$$

$$\int_{\partial V} * j = \int_\Sigma * j - \int_{\Sigma'} * j - \int_B * j$$

$$Q(\Sigma) = Q(\Sigma') + \underbrace{Q(B)}_{\to 0}$$

$$\underline{\underline{Q(\Sigma) = Q(\Sigma')}}$$

Homology classes of the surface

$$U_g = e^{i\lambda Q(\Sigma)}$$

Quantisation $Q(\Sigma) \rightarrow$ charge operator.

$\lambda \rightarrow$ Lie Algebra valued group action G.

$$g = e^{i\lambda} \qquad g \in G \quad \lambda \in \mathfrak{g}$$

$$\langle U_{g_1}(\Sigma)\, U_{g_2}(\Sigma) \rangle = \langle U_{g_1 g_2}(\Sigma) \rangle$$

$$\rightarrow \quad U_{g_1}(\Sigma)\, U_{g_2}(\Sigma) = e^{i\lambda_1 Q(\Sigma)}\, e^{i\lambda_2 Q(\Sigma)}$$

$$e^{X} e^{Y} = e^{Z} \qquad Z = X + Y + \tfrac{1}{2}[X, Y] + \cdots$$

$$\partial_\mu j^\mu_a(x)\, j^\nu_b(y) = f_{ab}{}^{c}\, j_c(x)\, \delta^{(d)}(x-y)$$

$$\langle d * j \rangle = 0$$

action of sym G.

$$\mathcal{O}_R(x) \longmapsto R(g)\, \mathcal{O}_R(x)$$

$$d * \langle j(x)\, \mathcal{O}_R(y) \rangle = \delta^{(d)}(x-y) \langle \widetilde{R(T^a)}\, \mathcal{O}_R(y) \rangle$$

$$\downarrow$$

$$\langle d * j(x)\, \mathcal{O}_R(y) \rangle = \langle \delta^{(d)}(x-y)\, R(T^a)\, \mathcal{O}_R(y) \rangle$$

$$U_g(\Sigma) = U_g(\Sigma')$$

$$\underline{U_g(\Sigma)} = e^{i\lambda Q(\Sigma)} = e^{i\lambda Q(\Sigma')} = \underline{U_g(\Sigma')}$$

$$U_{g_1} U_{g_2} = U_{g_1 g_2}$$ ← unitary operators

$$U_g(\Sigma)\, U_{g^{-1}}(\Sigma) = e^{i\lambda Q(\Sigma)}\, e^{-i\lambda Q(\Sigma)}$$
$$= e^{i(\lambda-\lambda) Q(\Sigma)}$$
$$= e^{0} = 1$$

$$g^{-1} = (g)^{-1}$$
$$= (e^{i\lambda})^{-1}$$
$$= e^{-i\lambda}$$

$$U_{g^{-1}}(\Sigma) = \left(U_g(\Sigma)\right)^{-1}$$

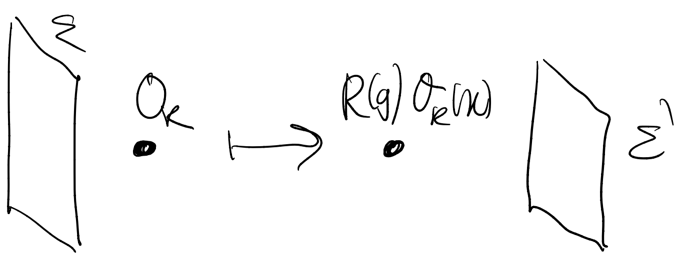
$j^{(1)} \to$ 1 form currents

$Q^{(0)} \to$ 0 form charges

$U_g(\Sigma) = e^{i\lambda Q(\Sigma)} \longrightarrow$ act on $O_R(x)$

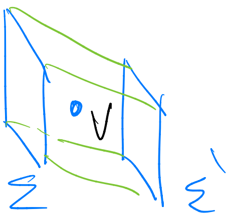
<u>Claim</u>

$$\left\langle U_g(\Sigma)\, O_R(x) \right\rangle = \left\langle R(g)\, O_R(x)\, U_g(\Sigma') \right\rangle$$



$$U_g(\Sigma)\, O_R(x)\, U_{g^{-1}}(\Sigma') = e^{i\lambda Q(\Sigma)}\, O_R(x)\, e^{-i\lambda Q(\Sigma')}$$

$$= e^{i\lambda Q(\Sigma)}\, e^{-i\lambda Q(\Sigma')}\, O_R(x) \qquad \langle \cdot \rangle$$



$$= e^{i\lambda (Q(\Sigma) - Q(\Sigma'))}\, O_R(x)$$

$$Q(\Sigma) - Q(\Sigma') = \int_\Sigma *j - \int_{\Sigma'} *j + \underbrace{\int_B *j}_{=0}$$

$$= \int_{\Sigma - \Sigma' + B} *j$$

$$= \int_V d*j$$

$$U_g(\Sigma)\, O_R(x)\, U_{g^{-1}}(\Sigma') = e^{i\lambda \int_V d*j}\, O_R(x)$$

$$= \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \left(\int_V d*j\right)^n O_R(x)\,.$$

$$d*\langle j(y)\, O_R(x)\rangle = *\delta^d(x-y)\, R(T^a)\, O_R(x)$$

$$\lambda = \lambda^a T^a$$

$$= \sum_{k=0}^{\infty} \frac{(i\lambda^a)^k}{k!} \left(\int_V d^d y \, \delta^{(d)}(x-y) \, R(T^a) \right)^k \mathcal{O}_R(x)$$

$$= \sum_{k=0}^{\infty} \frac{(i\lambda^a R(T^a))^k}{k!} \Big(\underbrace{\int_V d^d y \, \delta^{(d)}(x-y)}_{1} \Big)^k \mathcal{O}_R(x)$$

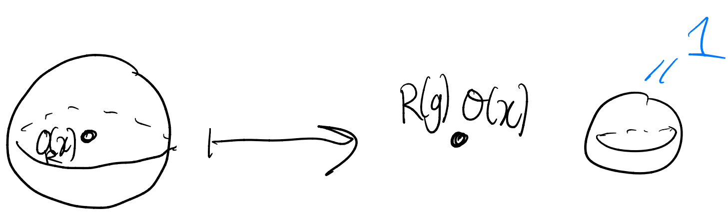
$$= \sum_{k=0}^{\infty} \frac{(i\lambda^a R(T^a))^k}{k!} \mathcal{O}_R(x)$$

$$= e^{\underbrace{i\lambda^a R(T^a)}_{R(\lambda)}} \mathcal{O}_R(x)$$

$$= R(g) \, \mathcal{O}_R(x)$$

$$U_g(\Sigma) \, \mathcal{O}_R(x) = R(g) \, \mathcal{O}_R(x) \, U_g(\Sigma)$$

<u>Corollary</u>



$$U_g(S^{d-1}) \, \mathcal{O}_R(x) = R(g) \, \mathcal{O}(x).$$

$G = U(1) \to$ electromagnetism.

$$O_q(x) \mapsto e^{i\lambda q} O_q(x)$$

Aharonov-Bohm effect.

Generalised Non-cts symmetries.

Lagrangian

$\Downarrow$

Noether currents $\Rightarrow$ Noether charges.

$\Downarrow$

$$U_g(\Sigma) = e^{i\lambda Q(\Sigma)}$$

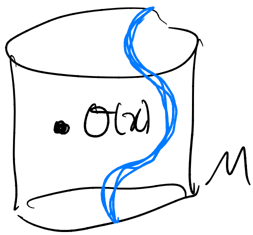
$\mapsto$ Discrete groups.
Non-Lagrange theories.

0-form sym: $Q^{(0)}$ $\quad j^{(1)}$ $\quad \Sigma^{(d-1)}$

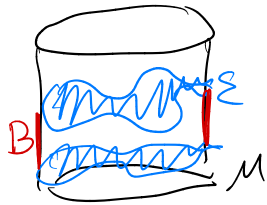
1-form sym: Q $\quad j^{(2)}$ $\quad \Sigma^{(d-2)}$

$\hookrightarrow$ Abelian.

Local operators



$$V_g(\Sigma) = e^{i\int_\Sigma *j}$$



$$Q(\Sigma) = Q(\Sigma') + Q(B)$$

$$e^{i\lambda_1 Q(\Sigma)}\, e^{i\lambda_2 Q(\Sigma)} = e^{Z}$$

$$Z = \lambda_1 Q + \lambda_2 Q + [\lambda_1 Q(\Sigma), \lambda_2 Q(\Sigma)] + \cdots$$

$$e^{i\lambda_1} e^{i\lambda_2} = e^{i\lambda_3}$$

$Q^a \to$ Generators

$$[\lambda_1^a Q^a(\Sigma), \lambda_2^b Q^b(\Sigma)] = \lambda_1^a \lambda_2^b [Q^a(\Sigma), Q^b(\Sigma)]$$

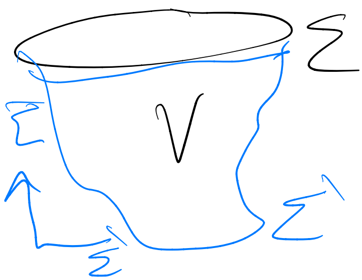
$$= \lambda_1^a \lambda_2^b f^{ab}{}_c\, Q^c(\Sigma)$$

$$= \lambda^c Q^c(\Sigma)$$

$$[Q^a(\Sigma), Q^b(\Sigma)] = \int_\Sigma \int_\Sigma [*j^a, *j^b]$$

$$\partial_\mu j^\mu_a(x)\, j_b^\nu(y) = f_{ab}{}^{c}\, j^\nu_c(x)\, \delta^{(d)}(x-y).$$

$$\int_\Sigma\int_\Sigma n_\nu n_\mu [j_a^\mu(x), j_b^\nu(y)]\, d^{d-1}x\, d^{d-1}y$$



$$[j^\mu_a(x), j^\nu_b(y)] = \underbrace{j^\mu_a(x)\, j^\nu_b(y)}_{\Sigma} - \underbrace{j^\nu_b(y)\, j^\mu_a(x)}_{\overline{\Sigma}}$$

$$\int_\Sigma \int_{\Sigma-\Sigma'} n_\nu n_\mu\, j^\mu_a(x)\, j^\nu_b(y)\, d^{d-1}x\, d^{d-1}y$$

$$= \int_\Sigma\int_\Sigma n_\nu n_\mu\, j^\mu_a j^\nu_b\, d^{d-1}x\, d^{d-1}y - \int_\Sigma\int_{\Sigma'} n_\nu n_\mu\, j^\nu_b j^\mu_a\, d^{d-1}x\, d^{d-1}y$$

$$= \int_\Sigma \int_{\Sigma-\Sigma'} n_\mu n_\nu\, j^\mu_a j^\nu_b\, d^{d-1}x\, d^{d-1}y$$

$$\left\langle = \int_\Sigma \int_V n_\nu \partial_\mu \left(j^\mu_a j^\nu_b \right) d^{d-1}x \, d^{d-1}y \right\rangle$$

$$= \int_\Sigma \int_V n_\nu f_{ab}{}^c j^\nu_c(x) \, \delta^d(x-y)$$

$$= \int_\Sigma n_\nu f_{ab}{}^c j^\nu_c(y)$$

$$= f_{ab}{}^c Q^c(\Sigma)$$

$$\therefore \left\langle [Q^a(\Sigma), Q^b(\Sigma)] \right\rangle = \left\langle f_{ab}{}^c Q^c(\Sigma) \right\rangle$$

$$\lambda Q = \lambda^a Q^a \qquad g = e^{i\lambda}$$

$$\boxed{\underline{\text{Then}} \quad U_{g_1}(\Sigma) U_{g_2}(\Sigma) = U_{g_1 g_2}(\Sigma)}$$