Chiral Anomalies & Non-invertible Syms.

Non-invertible arXiv: 2205.05086
symmetries in the SM.

Classical w/ Symmetries $\xrightarrow{\text{Quantisation}}$ Quantisation. symmetry "Broken" $\Rightarrow$ Anomaly.

$\psi, \bar{\psi}$ Dirac Spinors.

$$S = \int d^d x \mathcal{L} = \int dV \mathcal{L}$$

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi \underbrace{- m\bar{\psi}\psi}_{m=0}$$

Dirac Action/Lagrangian.

$$i\not{D} = i\gamma^\mu D_\mu \qquad D_\mu = \partial_\mu - igA_\mu \underbrace{+ \text{Spin-Connection}}$$

$$\gamma^\mu \rightarrow \eta^{\mu\nu} = (+,-,-,-)$$

Gamma Matrices in Minkowski space.

$$(\gamma^0)^2 = I \qquad (\gamma^i)^2 = -I$$

$$\gamma^\mu : \psi \mapsto \gamma^\mu \psi \qquad (\gamma^\mu)^\dagger = \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^i & \mu = i \end{cases}$$

$d=4$, η $\quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$(\gamma_5)^2 = I \qquad (\gamma_5)^\dagger = \gamma_5$$

$$\bar{\psi} = \psi^\dagger \gamma^0$$

Dirac

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/)\psi \qquad \alpha \in \mathbb{R}$$

$$U(1)_V : \quad \psi \mapsto e^{i\alpha}\psi \qquad \bar{\psi} \mapsto \bar{\psi} e^{-i\alpha}$$

Chiral

$$U(1)_A : \quad \psi \mapsto e^{i\alpha\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi} e^{i\alpha\gamma_5}$$

Euclidean

$$\bar{\psi} = \psi^\dagger \qquad (+,+,+,+) \qquad (\gamma^\mu)^2 = I$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} I$$

$$(\gamma^\mu)^\dagger = \gamma^\mu$$

Outline

1. Berezin integration

   Path integration $\int D\bar{\psi} D\psi \, e^{-S[\psi,\bar{\psi}]}$

   $\longrightarrow \int db \, da \, e^{-S[b,a]}$

2. Fujikawa Method — non-perturbative. Atiyah-Singer index thm.

$i\not{\partial}$ self-adjoint. Euclidean.

$$\langle \psi, \phi \rangle = \int dV \, \bar{\psi} \phi \rightarrow$$ positive definite inner-product.

Spectral Theorem

Hilbert space

$$H \rightarrow \underbrace{\psi}_{\text{Dirac field}} = \sum_k a_k \psi_k$$ $a \rightarrow$ Grassmann

ψ_k eigenspinor w/ eig. val λ_k $\quad i\not{\partial}\psi_k = \lambda_k \psi_k$

- $\psi_k \rightarrow$ complete basis (dense)
- $\lambda_k \rightarrow$ real

$$\bar{\psi} = \sum_k \underbrace{b_k}_{\text{Grassmann}} \bar{\psi}_k$$ $\bar{\psi}_k \rightarrow$ conjugate eigenspinors.

$$\langle \psi_k, \psi_j \rangle = \delta_{kj}$$ orthonormal

Rewrite the action in terms of Grassmann variables.

$$S = \int dV \, \bar{\psi} \underbrace{(i\not{\partial}) \psi}_{\lambda}$$

$$= \sum_{j,k} b_j a_k \lambda_k \underbrace{\int dV \, \bar{\psi}_j \psi_k}_{\delta_{jk}}$$

$$S = \sum_k (b_k a_k) \lambda_k$$

$$\int d\bar{\psi}\, d\psi$$

$$\psi = \sum_k a_k \psi_k = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ \vdots \end{bmatrix}$$

$$d^d x = d\underset{\sim}{x} = dx_1\, dx_2 \cdots dx_n$$

$$D\psi = \prod_k da_k$$

Measure

$$\int \underline{D\bar{\psi}}\, D\psi = \int \prod_k db_k\, da_k$$

$$Z = \int \prod_k \underline{db_k}\, da_k\, e^{-\sum_k (b_k a_k)\lambda_k}$$

$$\prod_k db_k = db_1 \cdots db_n \cdots$$

Grassmann $\{b_1, b_2\} = 0$

$$\langle \Phi \rangle = \frac{\int db\, da\, \Phi\, e^{-S}}{\int db\, da\, e^{-S}}$$

$$= \frac{-\int da\, db\, \Phi\, e^{-S}}{-\int da\, db\, e^{-S}}$$

Q: is there ambiguity with the ordering of the Grassmann numbers?

Answer: (prob) no, since divide by vacuum amplitude to get correlators!

Transformation under $U(1)_A$

$$da \mapsto da' \qquad \psi \mapsto e^{i\alpha\gamma_5}\psi = \psi'$$

$$\psi' = \sum_k a_k' \psi_k$$

$$\| $$

$$e^{i\alpha\gamma_5}\sum_k a_k \psi_k$$

$$\int \bar{\psi}_j \, dV \mapsto \sum_k \int dV \underbrace{\bar{\psi}_j \psi_k}_{\delta_{jk}} a_k' = \sum_k \int dV \bar{\psi}_j e^{i\alpha\gamma_5} \psi_k a_k$$

$$a_j' = \left(\sum_k \int dV \bar{\psi}_j e^{i\alpha\gamma_5}\psi_k\right) a_k$$

$$a_j' = \sum_k C_{jk} a_k \qquad C_{jk} = \int dV \bar{\psi}_j e^{i\alpha\gamma_5}\psi_k$$

$$b_j' = \sum_k C_{kj} b_k$$

$$a' = C a$$
$$b = C^T b$$

indep. of a, b

Measure

$$\int db\, da \xrightarrow{U(1)_A} \int db' \, da'$$

Measure $\int dy = \int \det(J)\, dx \qquad x \mapsto y \quad J_{ij} = \frac{\partial y_i}{\partial x_j}$

Berezin $\int d\underset{\sim}{a}' = \int (\det(J))^{-1} \, d\underset{\sim}{a}$

$$\int db' da' = \int db\, da\, \underbrace{\left(\det(C)\right)^{-2}}_{\text{Berezian}}$$

$$\frac{1}{\det C} = e^{-\mathrm{tr}\log(C)}$$ tr → over spinor indices

$$C_{jk} = \int dV\, \bar{\psi}_j\, e^{i\alpha\gamma_5}\psi_k = \delta_{jk} + \underbrace{i\int dV\, \alpha\, \bar{\psi}_j \gamma_5 \psi_k}_{} + O(\alpha^2)$$

$\alpha \to$ small

$$\log(1+x) = x + x^2 + \cdots$$

$$\Rightarrow\ C = I + X + O(\alpha^2)$$

$$\frac{1}{\det(C)} = e^{-\mathrm{tr}(X)}$$

$$\mathrm{tr}(X) = \mathrm{tr}\left(i\int dV\, \alpha\, \bar{\psi}_j \gamma_5 \psi_k\right)$$
$$= i\int dV \alpha \sum_k \bar{\psi}_k \gamma_5 \psi_k$$

$$\int \prod_k db_k' da_k' = \int \prod_k db_k da_k\, e^{-2\mathrm{tr}(X)}$$

$$= \int \prod_k db_k da_k\, e^{\underbrace{-2i\int dV\, \alpha \sum_k \bar{\psi}_k \gamma_5 \psi_k}_{\text{Anomaly term.}}}$$

$$\int d\bar{\psi}' d\psi' = \int d\bar{\psi}\, d\psi\, e^{-2i\int dV\, \alpha \sum_k \bar{\psi}_k \gamma_5 \psi_k}$$

- Vanish in dim = odd
- survives in dim = even

Exercise: show no anomaly term for $U(1)_V$

Fujikawa Method

$$\int dV \propto \sum_k \bar{\psi}_k \gamma_5 \psi_k \quad A$$

$$A = \sum_k \bar{\psi}_k \gamma_5 \psi_k \qquad \lambda_k \to \text{real.}$$

$$= \sum_k \bar{\psi}_k \gamma_5 \underbrace{\left(\lim_{M\to\infty} e^{-\left(\frac{\lambda_k}{M}\right)^2} \right)}_{\text{Fujikawa heat Kernel}} \psi_k$$

$$e^{-\left(\frac{\lambda_k}{M}\right)^2} \psi_k = e^{-\left(\frac{i\not{D}}{M}\right)^2} \psi_k$$

$$A = \lim_{M\to\infty} \sum_k \underbrace{\bar{\psi}_k}_{\text{FT}} \gamma_5 e^{-\left(\frac{i\not{D}}{M}\right)^2} \underbrace{\psi_k}_{\text{FT}}$$

$\tilde{\psi} \to \text{FT}$

identity: S some spinor operator.

$$\sum_j \bar{\tilde{\psi}}_j(k') \, S \, \tilde{\psi}_j(k) = (2\pi)^d \, \delta^{(d)}(k'-k) \, \mathrm{tr}(S)$$

$$A = \int \frac{d^dk}{(2\pi)^d} \, \mathrm{tr} \left(e^{-ik\cdot x} \gamma_5 \, e^{-\left(\frac{i\cancel{D}}{M}\right)^2} e^{ikx} \right)$$

trace over spinor indices. $\cancel{D}$
& gauge indices.

$$(i\cancel{D})^2 = -D^2 + \frac{ig}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}$$

here used $[D_\mu, D_\nu] = -ig F_{\mu\nu}$ $\quad$ $F \to$ curvature of Gauge Fields.

want to evaluate $e^{-\left(\frac{i\cancel{D}}{M}\right)^2} e^{ik\cdot x}$

$$e^{-ik\cdot x} D_\mu e^{ik\cdot x} = ik_\mu - igA_\mu$$

$$e^{-ik\cdot x} \underbrace{D_\mu D^\mu}_{D^2} e^{ik\cdot x} = -k^2 + O(A_\mu)$$

$$e^{-ik\cdot x} \frac{D_\mu D^\mu}{M^2} e^{ik\cdot x} = -\frac{k^2}{M^2} + O\left(\frac{A_\mu k^\mu}{M^2}\right)$$

$\frac{A_\mu}{M} \to 0$ as $M \to \infty$

$\frac{k}{M} \to$ survive.

$$\lim_{M\to\infty} e^{-ik\cdot x} e^{-\frac{D^2}{M^2}} e^{ik\cdot x} = \lim_{M\to\infty} e^{-\frac{k^2}{M^2}}$$

$$\lim_{M\to\infty} e^{-ik\cdot x} e^{-\left(\frac{i\not{D}}{M}\right)^2} e^{ik\cdot x} = \lim_{M\to\infty} e^{-ik\cdot x} e^{-\frac{D^2}{M^2} - \frac{ig}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} e^{ik\cdot x}$$

$$= \lim_{M\to\infty} e^{-\frac{k^2}{M^2} - \frac{ig}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}}$$

$$A = \sum_k \bar{\psi}_k \gamma_5 \psi_k$$

$$A = \lim_{M\to\infty} \mathrm{tr}\left(\gamma_5 e^{-\frac{ig}{2M^2}\gamma^\mu\gamma^\nu F_{\mu\nu}}\right) \int \frac{d^d\left(\frac{k}{M}\right)}{(2\pi)^d} e^{-\frac{k^2}{M^2}} \cdot M^d$$

- $\int_{\mathbb{R}^d} e^{-\|x\|^2} d^d x = \pi^{n/2}$

Gaussian integral in d-dims

$d=4$ Euclidean

$$\mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0$$

$$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4\,\varepsilon^{\mu\nu\rho\sigma}$$

Expanding exp.

$$\mathrm{tr}\left(\gamma_5 e^{\frac{-ig}{2M^2}\gamma^\mu\gamma^\nu F_{\mu\nu}}\right) M^4 = \underbrace{\cdots}_{=0 \text{ b/c tr}} - \frac{i^2 g^2 M^4}{2M^4}\mathrm{tr}\left(\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) + \underbrace{\frac{i g^3}{48 M^2}\mathrm{tr}(\cdots)}_{\to 0 \quad M\to\infty} + \cdots$$

$$A = \lim_{M\to\infty}\left(0 + \frac{g^2}{2}\mathrm{tr}\left(\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) + \mathcal{O}\left(\frac{1}{M^2}\right)\right)\frac{\pi^2}{(2\pi)^4}$$

$$A = \frac{g^2}{32\pi^2}\mathrm{tr}\left(\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)$$

$$\boxed{A = \frac{g^2}{8\pi^2}\mathrm{tr}\left(F\wedge F\right)} \longrightarrow d=4$$

$d=2k$ spacetime

$$A = \frac{g^k}{2^{3k-1}\pi^k}\frac{1}{(k-1)!}\mathrm{tr}\left(\varepsilon^{\mu_1\cdots\mu_{2k}}F_{\mu_1\mu_2}\cdots F_{\mu_{2k-1}\mu_{2k}}\right)$$

Measure ABJ Chiral transformation

$$\int d\bar{\psi}\, d\psi = \int d\bar{\psi}\, d\psi \exp\left(-\frac{iq^2}{4\pi^2}\int dV\, \alpha\, \mathrm{tr}(F\wedge F)\right)$$

Chiral Anomaly in $d=4$

variation classical action under cts sym $\alpha(x)$ (compact).

$$\delta S = -\int dV\, \alpha(x)\, \partial_\mu j^\mu$$

Ward identities (anomalous)

$$\boxed{d * \langle j \rangle = \frac{q^2}{4\pi^2}\langle \mathrm{tr}(F\wedge F)\rangle}$$

Recall "true" symmetry in $\rightarrow$ $d*\langle j\rangle = 0$
Quantum theory

Anomalies (ABJ) → Non-invertible symmetries

$$D_\alpha \otimes D_\beta = \gamma \, D_{\alpha\beta}$$

γ = fusion coefficient
= TQFT !

AWI

$$d * j = \frac{1}{4\pi^2} F \wedge F$$

<u>let us</u> work w/ Abelian Gauge theory

$F = dA$ $\quad A \to$ 1-form gauge field.

$$\frac{1}{4\pi^2} F \wedge F = \frac{1}{4\pi^2} dA \wedge dA = d(A \wedge dA) + A \wedge \underbrace{ddA}_{=0}$$

trivial cohomology.
Poincare lemma $d^2 \omega = 0$

$$d * j = \frac{1}{4\pi^2} d(A \wedge dA)$$

$$d\Big(\underbrace{* j - \frac{1}{4\pi^2} A \wedge dA}_{*j^A}\Big) = 0 \qquad d * j^A = 0$$

"conserved" Noether current. $\quad A \to$ gauge field.

<u>4d QED</u>

$*j^A \to$ codim-1 (=3 dim) $\quad \Sigma \to$ codim-1

Attempt

$$U_\alpha(\Sigma) = e^{i\alpha Q(\Sigma)} = e^{i\alpha\int_\Sigma\left(\star j - \frac{1}{4\pi^2} A\wedge dA\right)}$$

Even-dim $\Rightarrow$ Chiral Anomaly $\Rightarrow$ Noether current
$\Rightarrow$ Can construct Non-invertible syms.

Odd-dim $\Rightarrow$ Parity Anomaly $\Rightarrow$ Discrete
$\Rightarrow$ No (?) Generalised symmetry associated.

1hr 30min