Anomaly Inflow II

- Review Anomaly inflow Mechanism.
- 't Hooft anomalies → RG flow & Anomaly matching conditions.
- Example: particle on a circle

see arXiv: 1905.09315

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action

$$S=\int d^dx\, \mathcal{L}$$

Symmetry Group

$$G \times H$$

Global symmetry → Gauge symmetry

Manifold

$X \longrightarrow$ Principle bundle $\qquad G \to$ gauge group

insert background gauge field A

Ex. $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi \qquad U(1) \quad \phi \mapsto e^{i\alpha}\phi$

$$\mathcal{L} = (\partial_\mu + ig A_\mu)\phi^* (\partial_\mu - ig A_\mu)\phi$$

Anomalies

$G \times H$

$G \to$ gauge symmetry

$H \to$ Global symmetry.

(original) symmetry types (G)

	Gauge	Global
Gauge transformation G.	X *	't Hooft
(Mixed) Global transformation H	e.g. ABJ break H to a (discrete) subgroup.	't Hooft. e.g. Particle on a circle.

* 1. Anomaly inflow

2. Anomaly matching conditions $\Rightarrow$ e.g. SM.

Anomaly Ansatz $\Rightarrow$ e.g. Chiral Anomalies.

$$Z[A] \xmapsto[G,\, H]{} Z[A^\lambda] = Z[A] \exp\left(-2\pi i \int_X \alpha(\lambda, A)\right)$$

partition fnc

Anomalogy Cohomology

e.g. Regularisation schemes $\Rightarrow$ add counterterms to the action.

$\Rightarrow$ Partition fnc is subject to a regularisation scheme ambiguity.

$$Z[A] \sim Z[A] \exp\left(2\pi i \int_X \beta(A)\right)$$

$\beta(A) \Rightarrow$ counterterms.

Let the variation of β under $A \mapsto A^\lambda$ be : $\beta(A^\lambda) = \beta(A) - \delta_\lambda \beta(A)$

Then

RHS

$$Z[A^\lambda] = Z[A] \exp\left(-2\pi i \int_X \alpha(\lambda, A)\right) \exp\left(-2\pi i \int_X \delta\beta(A)\right)$$
$$\sim Z[A] \exp\left(-2\pi i \int_X \alpha(\lambda, A)\right)$$

LHS

$$\Rightarrow \quad \alpha \sim \alpha + \delta\beta$$

$$[\alpha] \in \{\alpha\} / (\alpha \sim \alpha + \delta\beta)$$

i.e. defines an anomaly cohomology.

Cohomology is discrete.

$\Rightarrow$ Anomaly equiv class is invariant under cts deformations of the theory S.

<u>Ex</u>
- Smoothly varying Lagrangian parameters
- Adding (infinitely) massive field.
- RG flow

$\hookrightarrow$ $[\alpha]$ invariant RG $\Rightarrow$ Anomaly matching

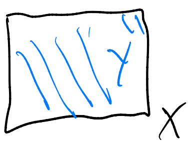
$$[\alpha]_{IR} = [\alpha]_{UV}$$

Anomaly Inflow Mechanism

Purposes

1. counter balance gauge anomalies.
2. Measure 't Hooft anomalies.

$$S = \int_X \mathcal{L} \qquad \partial Y = X \qquad A \Rightarrow \tilde{A} \qquad \tilde{A}|_X = A.$$



New bulk Lagrangian $\omega(\tilde{A})$

invertible field theory.

$$\mathcal{A}[\tilde{A}] = \exp\left(2\pi i \int_Y \omega(\tilde{A})\right)$$

1. want to construct.

2. transforms as

$$\mathcal{A}[\tilde{A}^\lambda] = \mathcal{A}[\tilde{A}] \exp\left(2\pi i \int_X \alpha(\lambda, A)\right)$$

$\Rightarrow$ Modified Partition fnc

Anomaly free.

$$\tilde{Z}[A] = Z[A]\,\mathcal{A}[\tilde{A}] \Rightarrow \boxed{\tilde{Z}[A^\lambda] = Z[A]}$$

Anomaly Matching

implication of anomaly cohomology:

smooth deformations $\omega \xmapsto{\mu} \omega_\mu$

$[\omega_\mu] \to$ equivalence classes.

invariant under · continuous variations
· RG flow
· etc.

$Z[A] \xrightarrow{\mu} Z_\mu[A]$ characterised by $[\omega_\mu]$

Claim if $[\omega_\mu] \neq 0 \Rightarrow Z_\mu[A]$ along all of μ will not invertible.

$\to$ Non-trivial theory.

Ex. RG.

$$\underset{\mu=0}{UV} \longrightarrow \underset{\mu=\infty}{IR}$$

Typically Expect IR to be a trivial theory No excitations eg. true vacuum.

$[\omega_\mu] \neq 0 \Rightarrow$
- Gapless: massless interacting CFT
- Gapped: nontrivial TQFT.

$G \times H \to$ Global symmetry

$\Rightarrow$ gauged $G \Rightarrow [\omega_\mu] \neq 0 \Rightarrow$ Know that the spectrum of the theory is non-trivial along RG flow.

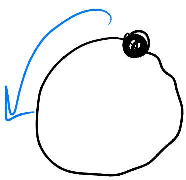
$$\begin{aligned} W_{\mu_1} &\to [W_{\mu_1}] \\ & \qquad \not\parallel \\ W_{\mu_2} &\to [W_{\mu_2}] \end{aligned} \Rightarrow$$ No way to deform one theory into the other.

Exception

$G \times H$ is spontaneously broken $\Rightarrow$ contain Goldstone modes
$\Rightarrow$ nontrivial gapless.

Key assumption to above analysis: $G \times H$ is preserved along flow μ.

Example Particle on a Circle



$$x(T) = x(0) \mod 2\pi$$
$$= x(0) + 2\pi n \quad n \in \mathbb{Z}.$$

Classical Lagrangian.

$$L = \underbrace{\frac{1}{2}\dot{x}^2}_{\text{Kinetic}} + \underbrace{\frac{\theta}{2\pi}\dot{x}}_{\text{Magnetic field.}}$$

Two symmetries

$U_\lambda : x \mapsto x - \lambda \qquad U(1)$

$C : x \mapsto -x \qquad \mathbb{Z}_2$

Want to study the 't Hooft of $U_\lambda \Rightarrow$ gauge sym.

$$L_0 = \frac{1}{2}(\dot{x} + A_0)^2 + \frac{\theta}{2\pi}(\dot{x} + A_0) \qquad A_0 \mapsto A_0 + \frac{d\lambda}{dt}$$

Quantum Theory

$$Z[A] = \int Dx \, e^{i \int dt \, L_0}$$

Wick Rotate $t \mapsto \tau$ $\Rightarrow$ Euclidean Theory

$$Z[A] = \int Dx \, e^{-\int d\tau \, L}$$ & add counterterm.

$$L = \frac{1}{2}(\dot{x} + A)^2 - i\frac{\theta}{2\pi}(\dot{x} + A) \underbrace{- ikA}_{\text{counterterm.}}$$

Decompose x into modes x_n w/ winding number n at $\tau = T$.

$$\Rightarrow \int Dx = \sum_{n \in \mathbb{Z}} \int Dx_n$$

$n \in \mathbb{Z} \to$ winding number of x

For simplicity, consider only $n=1$ contribution.

and can compactify domain

$$\int d\tau \, L = \oint d\tau \, L \qquad [0, T] \qquad x(T) = x(0) + 2\pi$$

consequences

1. $\theta \sim \theta + 2\pi$ $\quad$ [θ const.]
2. $k \in \mathbb{Z}$.

Proof 1

$$e^{-\frac{i}{2\pi}\oint(\theta+\phi)\dot{x}} = e^{-\frac{i}{2\pi}(\theta+\phi)\underbrace{(x(T)-x(0))}_{2\pi}} = e^{-i(\theta+\phi)}$$

Consider [invariant] shift of $\theta \mapsto \theta+\phi$

$\Rightarrow \phi = 2\pi$ s.t. theory is invariant.

Proof 2

$$e^{ik\oint A} \xrightarrow[\text{gauge transformation}]{A \mapsto A+\dot{\lambda}} e^{ik\oint A + ik\underbrace{\oint\dot{\lambda}}_{\underbrace{\lambda(T)-\lambda(0)}_{2\pi}}} = e^{ik\oint A}e^{ik2\pi}$$

$\Rightarrow k \in \mathbb{Z}$. For gauge invariance to hold.

L is gauge invariant.

- $\dot{x}+A \mapsto \dot{x}-\dot{\lambda}+A+\dot{\lambda} = \dot{x}+A$ ✓
- $kA \sim kA + k\dot{\lambda}$ since $k \in \mathbb{Z}$ ✓

What happens under C?

mixed 't Hooft anomaly

$$C: x \mapsto -x \qquad A \mapsto -A.$$

$\downarrow$

symmetry if $\theta = 0$ or $\theta = \pi$

$$L \xmapsto{C} L^C = \frac{1}{2}(\dot{x}+A)^2 + i\frac{\theta}{2\pi}(\dot{x}+A) + ikA$$

$$= \frac{1}{2}(\dot{x}+A)^2 - i\frac{\theta}{2\pi}(\dot{x}+A) - ikA + i\frac{\theta}{\pi}\dot{x} + i\frac{\theta}{\pi}A + 2ikA$$

$$= L + 2\frac{i\theta}{2\pi}\dot{x} + i\left(\frac{2\theta}{2\pi}+2k\right)A$$

$$e^{-\oint d\tau L} \mapsto e^{-\oint L} \, e^{-2i\frac{\theta}{2\pi}\oint d\tau \dot{x}} \, \underbrace{e^{-i\left(2k+\frac{2\theta}{2\pi}\right)\oint d\tau A}}_{e^{-2\pi i \int_X \alpha(\lambda, A)}}$$

$$\oint d\tau \, \dot{x} = x(T) - x(0) = 2\pi$$

$$e^{-2\pi i \int_X \alpha(\lambda, A)}, \quad \lambda \to c$$

$$e^{-2i\frac{\theta}{2\pi}\oint d\tau \dot{x}} = e^{-2i\theta} = 1 \quad \text{if} \quad \underline{\theta = 0, \pi}.$$

Symmetry.

<u>But</u> anomalous phase

$$e^{-i\left(2k+\frac{2\theta}{2\pi}\right)\oint d\tau \, A}$$

if $\theta = 0, \pi$ Then

$$Z[A] \mapsto Z[A^c] = Z[A] \exp\left(-i\left(2k+\frac{2\theta}{2\pi}\right)\oint d\tau A\right)$$

- $\theta = 0 \Rightarrow \left(2k+\frac{2\theta}{2\pi}\right) = 2k \Rightarrow$ invariant if $k=0$
- $\theta = \pi \Rightarrow 2k+\frac{2\theta}{2\pi} = 2k+1 \Rightarrow$ invariant if $k = -\frac{1}{2}$.

↳ gauge invariance requires $k \in \mathbb{Z}$!

A contradiction! i.e. a Quantum Anomaly.

Anomaly Inflow

$S^1 = X = \partial Y$

Extend A from boundary X to bulk Y.

$A \longrightarrow \tilde{A}$ s.t. $\tilde{A}\big|_X = A$.

$F = d(\tilde{A} d\tau)$ $(= d\tilde{A} \wedge d\tau)$

X = 1dim Manifold

Y = 2dim Manifold

needed s.t. $\exists$ curvature $F = dA$!

Note: on X, $d_X A = 0$

What is the bulk invertible field theory $\mathcal{A}[A]$ that cancels out the $\theta = \pi$ anomaly?

$$\exp\Big(-(2k+1)\, i \underbrace{\int_{S^1} (\tilde{A} d\tau)}_{\int_{\partial Y} (A d\tau) = \int_Y d(A d\tau) = \int_Y F}\Big) = \exp\left(-(2k+1)\, i \int_Y F\right)$$

Anomaly phase term.

$$\tilde{Z}[A] = Z[A] \exp\left(2\pi i \int_Y \omega(\tilde{A})\right)$$

$$\omega(\tilde{A}) = K F \qquad F \xmapsto{C} -F$$

Under the (anomalous) C transformation

$$\tilde{Z}[\tilde{A}^c] = Z[A^c] \exp\left(2\pi i \int_Y K(-F)\right)$$

$$= Z[A] \exp\left(-i(2k+1) \int_Y F\right) \exp\left(-2\pi i K \int_Y F\right)$$

$$= Z[A] \exp\left(2\pi i \int_Y K F\right)$$

$$\Rightarrow \quad 2k+1+K = -K$$

$$K = -\tfrac{1}{2}(2k+1)$$

$$\Rightarrow \quad \boxed{\tilde{Z}[\tilde{A}] = Z[A] \exp\left(-\tfrac{i}{2}(2k+1)\int_Y F\right)}$$

↳ w/ invertible field theory lagrangian $\omega = -\frac{1}{4\pi}(2k+1)F$

Thus

$\tilde{Z}[\tilde{A}]$ is C-invariant ! $\quad \tilde{Z}[\tilde{A}^c] = \tilde{Z}[\tilde{A}]$

Note: $Z[A] \to$ 1dim anomalous theory. $\quad \tilde{Z}[\tilde{A}] \to$ 2dim Anomaly free theory.

TODOs

- compute deformation class $[\omega_\mu]$ of "particle on a circle"
- compute anomaly cohomology of "particle on a circle"
- understand anomaly inflow picture for discrete symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ (shift symmetry broken to discrete gauge group ↓ $\mathbb{Z}_2$; ↑ C for the second $\mathbb{Z}_2$)

e.g. ↳ turned on by adding $\cos(2x)$ to Lagrangian ⇒ expect to be in the same deformation as analysis above.

- RG flow of "particle on a circle". [& repeat above analysis]
- What if? Gauge C & consider global (non-gauge) shift symmetries? ↗